NASA-CR-203504

711-64-CR
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Using Critical Point Theory andLocal Differential Geometry to Visualize Scalar Fields Part I: Two Dimensional Data

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Report RNR-89-007, November 1989

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Abstract. Tools adapted from the local differential geometry of surfaces are used to aid visualization of scalar fields in 2-D by means of a global embedding in flat 3-D space. By these means, "skeletal" information about flowfield quantities can be deduced and thus circumvent the necessity of drawing very large numbers of graphical objects.

[1]This work was supported by NASA Contract NAS 2-12961to Computer Sciences Corporation for the NAS Applied Research Office at NASA Ames Research Center.

November 24, 1989

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Using Critical Point Theory and Local Differential Geometry to Visualize Scalar Fields Part I: Two Dimensional Data Report RNR-89-007

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1 Abstract

Visualizations of scalar fields in two dimensions can be enhanced by creating a "height map" embedded in three dimensions, applying critical point theory and the local differential geometry of surfaces to this height map, and then projecting back down into two dimensions.

The critical points of the scalar field are determined and classified using the first and second derivatives of the function which defines the height map. From these derivatives, the first and second fundamental forms of the surface are computed, by which the scalar field can be partitioned into regions of positive, negative, and zero curvature. Features like ridge lines and trough lines are obtained by integrating the lines of curvature between the local maxima and minima. The graphical representation of these lines provides a "morphological skeleton" which can be rendered with fewer graphical objects than a full set of isosurfaces or contour lines. Furthermore, this representation provides a means by which different fields can be compared by type.

The computer program FCRIT2 encodes these ideas in the C programming language using the NASA Ames Panel Library [TRIS89] on Silicon Graphics workstations running the UNIX operating system.

2 Introduction: Visualizing Scalar Fields

The most common type of multidimensional data in scientific visualization is the scalar field. A scalar field is a function of space which is characterized completely by its magnitude, that is, by a single dependent variable. Thus $f(x, y)$ represents a scalar field in 2-D.

2.1 Visualization Techniques

Techniques for visualizing scalar fields are discussed in [UPSO89] and references cited therein. The most common method for representing a scalar field $f(x, y)$ in two dimensions is by means of a contour plot, i.e. a plot of the curves at which the field takes on a set predetermined values. If color equipment is available, one can assign a color to the contours based on the value of the scalar. Plate 1 shows a PLOT3D [PLOT89] contour plot of test function, used throughout this paper, which is a sum of trigonometric functions. Color is one of several types of *transfer function* which maps a physical parameter into a visual one. The color scale chosen here and hereafter is borrowed from PLOT3D and uses fully saturated colors with blue and cyan representing low values and red and magenta representing high values.

Depending on the choice of these values and their accuracy, one can form a mental picture of the field, including its local minima, maxima, and saddle points. One disadvantage of contour maps is that information can be missed unless the contours are chosen judiciously.

If one takes the limit of infinitely many colored contours, one obtains a smooth-shaded plot. These plots are becoming increasingly common as color interpolation hardware becomes standard on graphics workstations.

If three-dimensional graphics equipment is available, one can create a "height map" $z(x, y)$ proportional to the scalar field $f(x, y)$ or to some monotonic function of f such as $log(f)$ or $argsinh(f)$. The mapping $f \rightarrow z$ is a *geometric transfer function* relating a physical scalar quantity to a geometric one (height). This height map can be used directly in the visualising the scalar field in 3-D using lighting, shading, and depth-cueing as appropriate. Plate 2 shows an FCRIT2 plot of a "wire frame" height map together with its projection on the x-y plane.

Alternatively, one can use a 3-D lighting model, and then project the results of lighting operations back down into 2-D. The topographical maps which use shading to indicate mountain ranges use a variant of this technique.

2.2 Present Method

In the method outlined in this paper, we will create a height map using the third dimension as a transfer function. The we will apply the tools of classical differential geometry to find the significant features of the map. Optionally, we

will project these features back down to the original space. Thus we will develop an automated method for extracting significant structures in the data.

An important reason for the projection back down is that it allows an extension of these ideas to 3-D scalar fields where one would use a 4-D height map projected back down to 3-D. These ideas will be developed more fully in a subsequent publication.

3 Critical Point Analysis of 2D Scalar Fields

The *critical points* of a function $f(x)$ in one dimension are the points at which f or any of its derivatives vanish. The most important of these points are the seroes of f and the seroes of df/dx The sign of the second derivative is then used to classify thse points as minima ($d^2f/dx^2 > 0$), maxima($d^2f/dx^2 < 0$), or points of inflection ($d^2f/dx^2 = 0$).

In two dimensions, one looks at the seroes of $f(x, y)$, and the seroes of its gradient ∇f. Combinations of second partial derivatives (curvatures of various types) are then used to classify these points. In dimensions two and higher, the curvatures solve an eigenvalue problem, whose eigenvectors, when integrated, are *lines of curvature* of the surface.

3.1 Zeroes

The set of seroes of a 2-D function are just the contour lines corresponding to $F(x, y) = 0$. and can be found by using any convenient contouring routine. In FCRIT2, we have used a table-lookup scheme in the style of "marching cubes" [LORE86]. This contour curve can be combined with other curves in the context of an interactive tool.

3.2 Gradient

The first derivatives of a scalar function form a vector called the *gradient*

$$\nabla f = (f_1, f_2)$$

Here, the notation f_1 means the partial derivative of f with respect to its first argument, i.e. $f_1 = f_x = \partial f/\partial x$.

The gradient function is a vector field, and can be visualised using arrows. It is visually effective to combine a 2-D plot of the gradient with contour curves of its square. Plate 3 shows an FCRIT2 vector plot of the gradient in 2-D and 3-D with vector length proportional to magnitude of the gradient as well as coloring by the same quantity.

The seroes of the gradient are also the seroes of its square magnitude. Because seroes of the gradient often are isolated points, and also because of the discretisation errors which may be involved in computing the derivatives, it may

be more useful to plot the contour surface corresponding to some small positive value ϵ of $(\nabla f)^2$. Plate 4 shows the seroes of the $f(x, y)$ plotted in green together with the seroes of the gradient plotted in yellow. The seroes of f form a continuous curve which is here approximated by its intersections with grid lines. Anticipating the next section, the minima are tagged with text in blue and the maxima in red.

4 Local Differential Geometry

Further investigation of critical points requires evaluation of some second derivative quantities. For the sake of what follows, we will now introduce some terminology from local differential (Riemannian) geometry. The standard analysis of the surface $\mathbf{X}(x, y)$ can be outlined as follows. For more details see for example [STOK69]. We follow some of his notational conventions in the following. [SCHO54] is a good reference for higher dimensional generalisations of these ideas.

Again let $f(x, y)$ be a scalar function in Euclidean two-dimensional space E^2, here called the *base space*. Then we can represent this function by a 2-D surface embedded in E^3

$$\mathbf{X}(x, y) = (x, y, f(x, y))$$

Boldface type in this section indicates a vector in E^3. We assume that the transfer function $f(x, y)$ has been suitably scaled so that its non-dimensionalized value is comparable to the other two dimensions.

Results derived in this section are expressed as tensors which are valid in any local curvilinear coordinate system x^i but we will continue to use the rectangular coordinates $(x^1, x^2) = (x, y)$ since we have them at our disposal, and we will need components with respect to this system for visualisation purposes.

4.1 Metric Tensor

The square of the element of length along the surface of the height map $\mathbf{X}(x, y)$ is

$$ds^2 = dx^2 + dy^2 + df^2$$

The first two terms represent the length element in the base space and df is the change in the height function

$$df = f_1 dx + f_2 dy$$

Therefore,

$$ds^2 = (1 + f_1^2)dx^2 + (1 + f_2^2)dy^2 + 2f_1 f_2 dx dy.$$

This can also be written in terms of the differential of $\mathbf{X}$ as a scalar product

$$ds^2 = d\mathbf{X} \cdot d\mathbf{X}.$$

Here,

$$d\mathbf{X} = \mathbf{X}_1 dx + \mathbf{X}_2 dy$$

and

$$\mathbf{X}_1 = \frac{\partial \mathbf{X}}{\partial x} = (1, 0, f_1), \qquad \mathbf{X}_2 = \frac{\partial \mathbf{X}}{\partial y} = (1, 0, f_2)$$

The vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ at any point span a two dimensional vector space called the *tangent plane* (in N dimensions the *tangent space*) and form a coordinate basis for that space. Using an index notation with the convention that repeated indices are summed, one defines the components of the *metric tensor* g_{ij} by

$$ds^2 = g_{ij} dx^i dx^j, \qquad i, j = 1, 2$$

$$g_{ij} = \mathbf{X}_i \cdot \mathbf{X}_j = \delta_{ij} + f_i f_j$$

The symbol δ_{ij} is the Kronecker delta which equals one when $i = j$ and otherwise vanishes. One can verify that the metric components are

$$g_{ij} = \begin{bmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{bmatrix}$$

The determinant of metric tensor

$$g = 1 + f_1^2 + f_2^2 = 1 + |\nabla f|^2$$

is positive definite and has a minimum where $|\nabla f| = 0$ Its square root represents the ratio of an area on the height surface to the projected area in the base space.

The metric tensor and its derivatives contain all the information we need to determine the local properties of the surface. The inverse (contravariant) metric tensor is defined by

$$g^{ij} g_{jk} = \delta^i_k$$

so that in 2-D

$$g^{ij} = \frac{1}{g} \begin{bmatrix} 1 + f_2^2 & -f_1 f_2 \\ -f_1 f_2 & 1 + f_1^2 \end{bmatrix}$$

that is,

$$g^{11} = g_{22}/g \qquad g^{12} = -g_{12}/g \qquad g^{22} = g_{11}/g$$

A further notational convention is that indices are raised and lowered by multiplication by the covariant and contravariant forms, respectively, of the metric tensor, followed by contraction on (summation over) the repeated indices. Under this convention, pairs of indices occur only with one index raised, the other lowered. Dots are used as position-holders in tensors which have both kinds of indices, since index position is significant.

4.1.1 Visualisation

At this point, we have defined an auxiliary vector field ∇f, the tensor field g_{ij} and scalar field g (which is just the square of the gradient plus one). The scalar field can be visualised by the usual methods of contour plotting, and the gradient field is usefully represented by arrows. There are visualisation methods available for tensor fields, but these are more appropriately discussed after we have introduced curvature. Of course, on can produce carpet plots of the various components, but they are generally hard to interpret.

4.2 Orthonormal Triad

Consider a generic point on the surface at $\mathbf{X}_0 = (x_0, y_0, f(x_0, y_0))$. The vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ span the tangent plane at $\mathbf{X}_0$. The unit normal to the tangent plane, and thus to the surface, is found by taking the vector product of $\mathbf{X}_1$ and $\mathbf{X}_2$ and normalizing:

$$\hat{\mathbf{N}} = (-f_1, -f_2, 1)/\sqrt{g}$$

Note that the vectors $\mathbf{X}_1, \mathbf{X}_2$ are not orthogonal in 3-space. One can derive an orthonormal triad of vectors using the Gram-Schmidt process. One such orthonormal basis can be constructed by starting with $\mathbf{X}_1$ and is given by

$$\begin{aligned}
\hat{\mathbf{e}}_1 &= (1, 0, fx)/\sqrt{g_{11}} \\
\hat{\mathbf{e}}_2 &= (-f_x f_y, g_{11}, f_y)/(\sqrt{g}\sqrt{g_{11}}) \\
\hat{\mathbf{e}}_3 &= (-f_1, -f_2, 1)/\sqrt{g} = \hat{\mathbf{N}}
\end{aligned}$$

One can easily verify that they form an orthonormal basis by looking at their scalar and vector products. This set of vectors can then be visualised in 3-D. Plate 5 shows the normal vectors as they appear on the height map, together with their projection down into 2D. The projection of the entire triad can also be performed, but since there is no invariant meaning to their choice within the tangent plane, their visualisation adds no essential information and tends to produce a confusing picture.

With some practice, one can get a feel for the properties of the height map from looking at the projection. While not important here, such a visualisation may be useful in the case of one dimension higher (4-D height maps of 3-D scalar fields) which will be the subject of a subsequent paper.

4.3 Second Derivatives and Curvature

In the critical point analysis of a 1-D function, one examines the sign of the second derivatives at the zeroes of the first derivatives. One also finds the *points of inflection* which are zeroes of the second derivative. In 2-D, the analogous

process involves the study of curvature. We make use of the second partial derivatives of f to analyse the deviation between the surface and its tangent plane. Taylor expansion of $\mathbf{X}$ $\mathbf{X}_0$ yields

$$\mathbf{X}(x,y) = \mathbf{X}_0 + \mathbf{X}_1 x + \mathbf{X}_2 y + \frac{1}{2}[\mathbf{X}_{11}x^2 + 2\mathbf{X}_{12}xy + \mathbf{X}_{22}y^2] + O(x^3).$$

where the coefficients of the right hand side are evaluated at (x_0, y_0), and

$$\mathbf{X}_{ij} = \frac{\partial \mathbf{X}}{\partial x^i \partial x^j} = (0, 0, f_{ij})$$

The height of the surface above its tangent plane is

$$h(x,y) = [\mathbf{X}(x,y) - \mathbf{X}(0,0)] \cdot \hat{\mathbf{N}}$$

The linear terms in this expression vanish, and the leading term which remains is quadratic in the base-space coordinates:

$$h(x,y) = \frac{1}{2}[b_{11}x^2 + 2b_{12}xy + b_{22}y^2] + O(x^3).$$

The leading term in $h(x,y)$ is called the *second fundamental form*

$$II = \frac{1}{2} b_{ij} x^i x^j.$$

It defines a quadric called *osculating quadric* to the surface. The coefficients b_{ij} are the covariant components of the *extrinsic curvature tensor* are given in terms of f by

$$b_{ij} = \mathbf{X}_{ij} \cdot \hat{\mathbf{N}} = f_{ij}/\sqrt{g}$$

The determinant of this tensor is given by

$$|b_{ij}| = (f_{11}f_{22} - f_{12}^2)/g$$

The numerator is the determinant of the *Hessian matrix* of f, defined by

$$Hess(x,y) = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix}$$

In two dimensions, the sign of the Hessian determinant determines the characteristics of the surface at $\mathbf{X}_0$.

4.4 2-D Curvature Eigenvalue Problem

The curvature of the surface is analysed by studying the *eigenvalue problem for curvature*. That is, we wish to find the basis of vectors v^i for which

$$(b_{ij} - kg_{ij})v^i = 0$$

The standard form of the eigenvalue problem is obtained by multiplying both sides of the preceding equation by g^{jk}. Since the "mixed" form of the metric tensor is diagonal (the Kronecker delta) we obtain the form

$$\begin{bmatrix} b_1^{\cdot 1} - k^{(n)} & b_1^{\cdot 2} \\ b_2^{\cdot 1} & b_2^{\cdot 2} - k^{(n)} \end{bmatrix} \begin{bmatrix} v_1^{(n)} \\ v_2^{(n)} \end{bmatrix} = 0$$

One finds the eigenvalues k by setting the determinant equal of this equation to zero, obtaining thereby the *characteristic equation*

$$k^2 - k(b_1^{\cdot 1} + b_2^{\cdot 2}) + (b_1^{\cdot 1} b_2^{\cdot 2} - b_1^{\cdot 2} b_2^{\cdot 1}) = 0$$

The formulae for the mixed components $b_i^{\cdot j}$ are as follows:

$$\begin{aligned}
b_1^{\cdot 1} &= g^{-\frac{3}{2}}[(1+f_2^2)f_{11} - f_1 f_2 f_{12}] \\
b_1^{\cdot 2} &= g^{-\frac{3}{2}}[(1+f_1^2)f_{12} - f_1 f_2 f_{11}] \\
b_2^{\cdot 1} &= g^{-\frac{3}{2}}[(1+f_2^2)f_{12} - f_1 f_2 f_{22}] \\
b_2^{\cdot 2} &= g^{-\frac{3}{2}}[(1+f_1^2)f_{22} - f_1 f_2 f_{12}]
\end{aligned}$$

The characteristic equation has two real solutions, the eigenvalues k_1, k_2 which are called *principal curvatures*. They are the maximum and minimum curvatures of normal sections of the surface. The characteristic equation can be rewritten in the following suggestive form:

$$k^2 - 2Hk + \kappa = 0$$

which has solutions

$$k_1 = H + \sqrt{H^2 - \kappa} \qquad k_2 = H - \sqrt{H^2 - \kappa}$$

The *Mean curvature* H is the average of the principal curvatures

$$H = \frac{k_1 + k_2}{2} = \frac{b_1^{\cdot 1} + b_2^{\cdot 2}}{2}$$

In terms of the derivatives of f:

$$H = \frac{[f_{11}(1+f_2^2) + f_{22}(1+f_1^2) - 2f_1 f_2 f_{12}]}{2(1+f_1^2+f_2^2)^{\frac{3}{2}}}$$

The *Gaussian curvature* κ is the product of the principal curvatures, and is also the ratio of the determinants or the second and first fundamental forms:

$$\kappa = k_1 k_2 \qquad = \frac{|b_{ij}|}{|g_{ij}|}$$

In terms of the derivatives of f:

$$\kappa = \frac{f_{11}f_{22} - f_{12}^2}{(1 + f_1^2 + f_2^2)^2}$$

The determinant of g is positive definite. Therefore, the sign of κ is the same as the sign of sign of $|b_{ij}|$ as well as of the Hessian. A *minimum* has k_1, k_2 both positive; a *maximum* has k_1, k_2 both negative. They are both *elliptic points* with positive Gauss curvature. A saddle or *hyperbolic point* has principal curvatures of opposite signs, and therefore negative Gauss curvature. a parabolic point has one curvature vanishing, and a planar region has both of them vanishing. Parabolic and planar points have vanishing Gauss curvature. Since the curvature is a continuous function of a differentiable surface, elliptic and hyperbolic regions must be separated by curves or regions consisting of parabolic or planar points.

Plate 6 is a plot of the Hessian determinant. The coloration is chosen so that cyan represents points with a negative Hessian. Points with a positive Hessian are colored blue or red, repsectively, depending upon whether the sign of the principal curvatures are both negative or both positive (i.e. according to the mean curvature).

By offsetting the tangent plane slightly along the normal, one obtains a characteristic figure as the intersection of the plane and the surface. This figure is called the *Dupin indicatrix*. For an elliptic point it is an ellipse, for an hyperbolic point it is a hyperbola. It is a pair of lines (degenerate conic) at a parabolic point, and it vanishes at a planar point. Todd [TODD86] has used an iconic visual representation of these indicatrices in a biological study. This would allow one to abbreviate Plate 6 into a set of bounding parabolic curves, with each region identified by such an icon. This graphical representation is alas not yet supported in the FCRIT2 software.

5 Lines of Curvature

5.1 Eigenvectors

Corresponding to these curvatures are two orthogonal directions (the eigenvectors $v^{(1)}, v^{(2)}$) which are called the *principal directions*. Note that the eigenvectors are defined only up to a multiplicative constant.

The unit eigenvectors corresponding to k_1 and k_2, respectively, are

$$v^{(1)} = (b_1^{\cdot 2}, k_1 - b_1^{\cdot 1})/\sqrt{(b_1^{\cdot 2})^2 + (b_1^{\cdot 1} - k_1)^2}$$

$$v^{(2)} = (b_1^{\cdot 2}, k_2 - b_1^{\cdot 1})/\sqrt{(b_1^{\cdot 2})^2 + (b_1^{\cdot 1} - k_2)^2}$$

One can verify that these eigenvectors are orthogonal. The set of eigenvectors can be visualised directly in 3-space and also they can be projected into 2-space,

as shown in Plate 7. A more informative plot is shown in Plate 8, where vectors are plotted whose lengths correspond to the principal curvature associated with that eigenvalue. The regions of strong curvature are more strongly indicated, especially in the 2-D projection.

5.2 Integration

The two mutually orthogonal vector fields of principal directions can be integrated to produce the *lines of curvature* of the surface. Since FCRIT2 does not yet have this capability, 2 separate runs of a modified version of PLOT3D were mad to plot the integral curves of $v^{(1)}$ and $v^{(2)}$, respectively, and are shown nin Plates 9 and 10. The color of the curve gives the function value, and one can see that the curve corresponding to $v^{(1)}$ corresponds to the "ridge-like" curves and that the integral curves of $v^{(2)}$ are the "trough-like" curves. Some software by Robert Dickinson [DICK89] can interactively integrate the eigenvectors of symmetric tensor fields, and can therefore be applied to this problem.

The most significant of these curves are those which emanate from the critical points. Plates 9 and 10 were obtained using PLOT3D, which restricts initial rakes to grid points. The next development, which time did not permit, will be to integrate the particle trace integration with the rest of the FCRIT software in order to permit the generation of lines of curvature which originate and terminate at crital points only.

5.3 Schematization

We now have the tools at hand to produce a "morphological skeleton" of a scalar field. Thus, the procedure to produce and visualise a morphological skeleton of the scalar field f as follows:

1. Plot the contours $f = 0$ and $\kappa = 0$.
2. Find the critical points $|\nabla f| = 0$.
3. For each of the critical points,find the eigenvalues of curvature and classify the point as a minimum, maximum, saddle, or inflection point.
4. Solve the curvature eigenvalue problem at critical points.
5. integrate the lines of curvature between them, computing additional principal directions as needed (or pre-compute these directions and store them).

Then, the aim of the paper will be accomplished, which is to represent the scalar field by a small set of graphiucal objects, namely the zeros of the function and its gradient, and selected lines of curvature.

6 Future Developments

6.1 Finite Differences and Truncation Error

In CFD, one represents both grids and solutions by finite collections of points, thereby necessarily introducing em truncation errors. For example, one finite representation of f_{xx} is

$$f_{xx} \frac{f(x+\Delta x)-f(x-\Delta x)}{\Delta x^2} + \frac{1}{12} f_{xxxx}(\Delta x)^2$$

Ultimately, it will be desirable to include estimates of these errors in the analysis, as the truncation errors of higher derivatives are in general less acurate than the function evaluations. In the present work we have avoided this problem by using a known analytic function. For real data, these issues will have to be addressed.

6.2 Curvilinear Coordinates

CFD codes for aerodynamic bodies use *grids* based on curvilinear coordinates.

$$x = x(\xi, \eta) \qquad y = y(\xi, \eta)$$

This adds to the complexity of coding, and additional truncation error introduced by the finite difference approximation to the map.

The ideas developed previously are still valid, since they are based on geometrical invariants which do not depend on coordinates, but their expression is more complicated. For example, the vectors of derivatives with respect to ξ and η are as follows:

$$\mathbf{X}_\xi = (x_\xi, y_\xi); \qquad \mathbf{X}_\eta = (x_\eta, y_\eta); \qquad \mathbf{X}_{\xi\eta} = (x_{\xi\eta}, y_{\xi\eta}), \qquad etc.$$

The *Jacobian matrix* is given by

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \qquad = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$$

and relats the first derivatives in the respective coordinate systems. Then, one can express the derivative of a scalar with respect to rectangular coordinates by using the chain rule, as in

$$f_x = \xi_x f_\xi + \eta_x f_\eta$$

Second derivatives can be derived by chain rules as well, but have expressions involving "second Jacobians" which are too complicated to reproduce here.

6.3 Three-Dimensional Data

Many of the ideas of local differential geometry can be carried out for three-dimensional fields. The basic idea will be to create a 4-dimensional "height map" above the 3-D scalar field and then to apply analogous techniques. However, since we do not have much ability to present 4-D objects on a workstation, we will have to depend much more on the projections into 3-D which are analogous to the 2-D plots in the present work.

The seroes of the function are still isolated points. However, the serioes of the gradient are now in general surfaces. These "parabolic surfaces" will divide the space into regions characterised by the eigenvalues of the extrinsic curvature matrix. This is now a 3 by 3 matrix, so there are now eight possible permutations of signs which correspond to positive and negative Gaussian curvature. Besides the Gaussian and Mean curvatures, another useful quantity is the "mean radius of curvature." These three quantities are the coefficients of the characteristic equation which determine the type of region. Visualisation of these regions is still an open question. One promising idea, extending Todd's method, is the use of 3-D icons one to each region, to represent the Dupin indicatrix, which be a type of quadric surface.

7 Summary

A scheme has been developed, and partially implemented, to create a visual abbreviation or shorthand for a scalar field in two dimensions. Such an shorthand notation will, it is hoped, provide a useful tool for scientists interested in classifying and comparing scalar fields which arise in numerical simulations of or experiments measuring physical phenomena.

8 References

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9 Figure Captions

Plate 1. PLOT3D contour plot of $f(x,y)$.

Plate 2. FCRIT2 plot of function and height surface.

Plate 3. FCRIT2 plot of gradient field. Vector length proportional to magnitude of gradient, color transfer function also based on magnitude of gradient.

Plate 4. FCRIT2 plot of zeroes of f and ∇f.

Plate 5. FCRIT2 plot of unit normals to height surface and projection into 2-D.

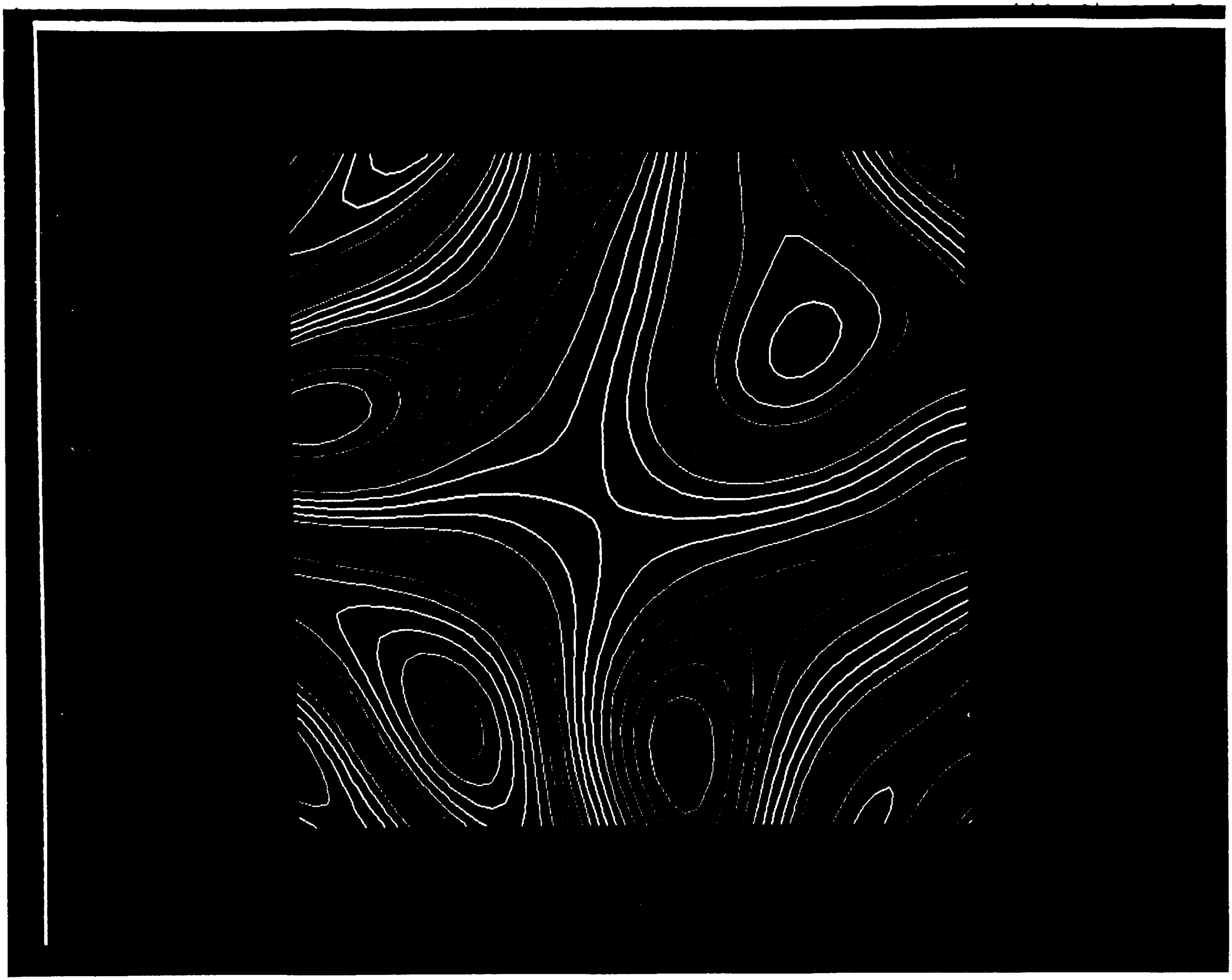
Plate 6. FCRIT2 carpet plot of Hessian determinant of $f(x,y)$.

Plate 7. FCRIT2 plot of unit curvature eigenvectors. Note that these vectors are orthogonal in the tangent planes to the height surface.

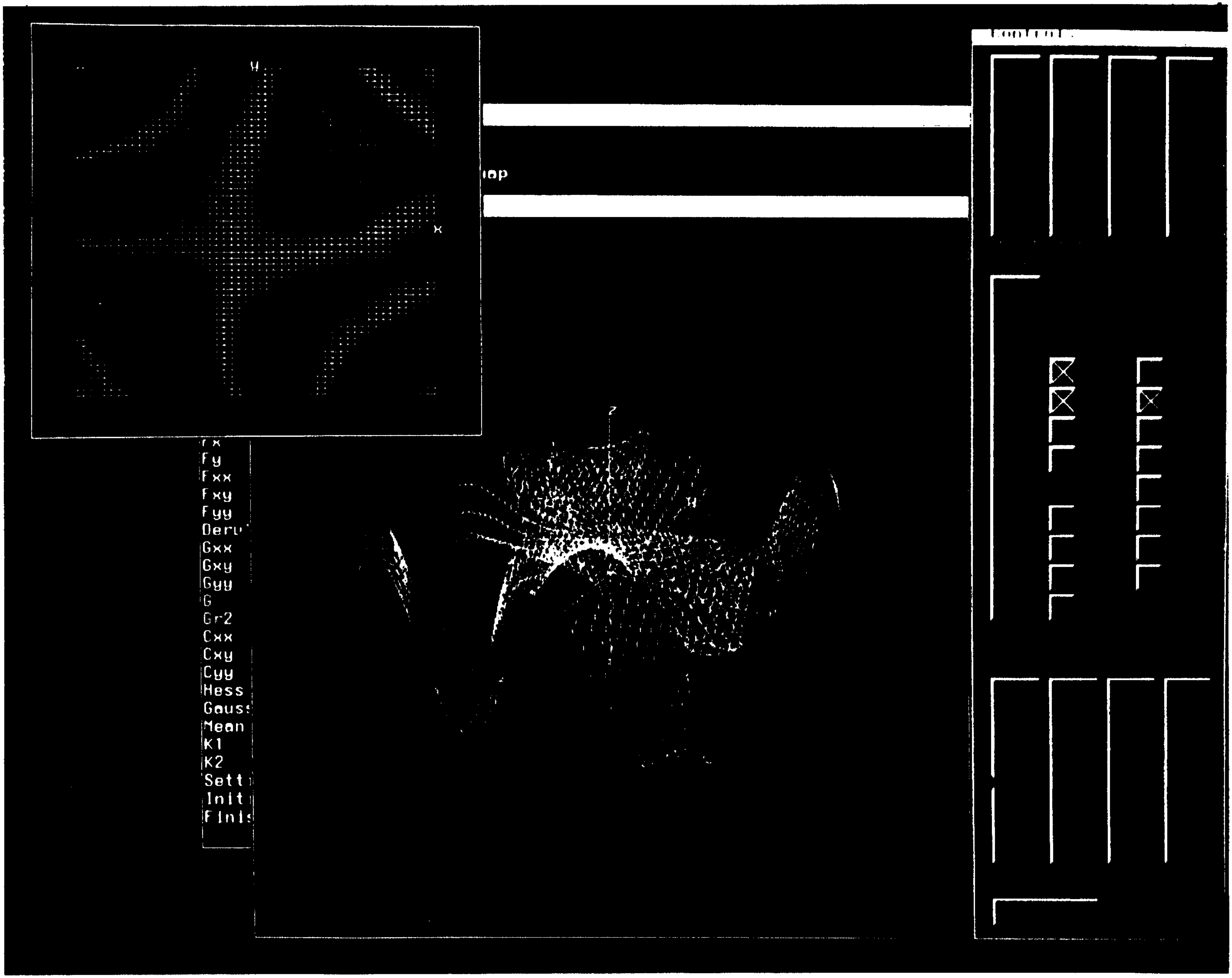
Plate 8. FCRIT2 plot of curvature eigenvectors, with lengths proportional to the magnitude of the corresponding eigenvalue.

Plate 9. PLOT3D plot of the integral curves of the first curvature eigenvector field, including "ridge lines."

Plate 10. PLOT3D plot of the integral curves of the second curvature eigenvector, showing "trough lines."



PLOT3D 2D contour plot of function KERLICK FCRIT2 Plate 1


Fy
Fxx
Fxy
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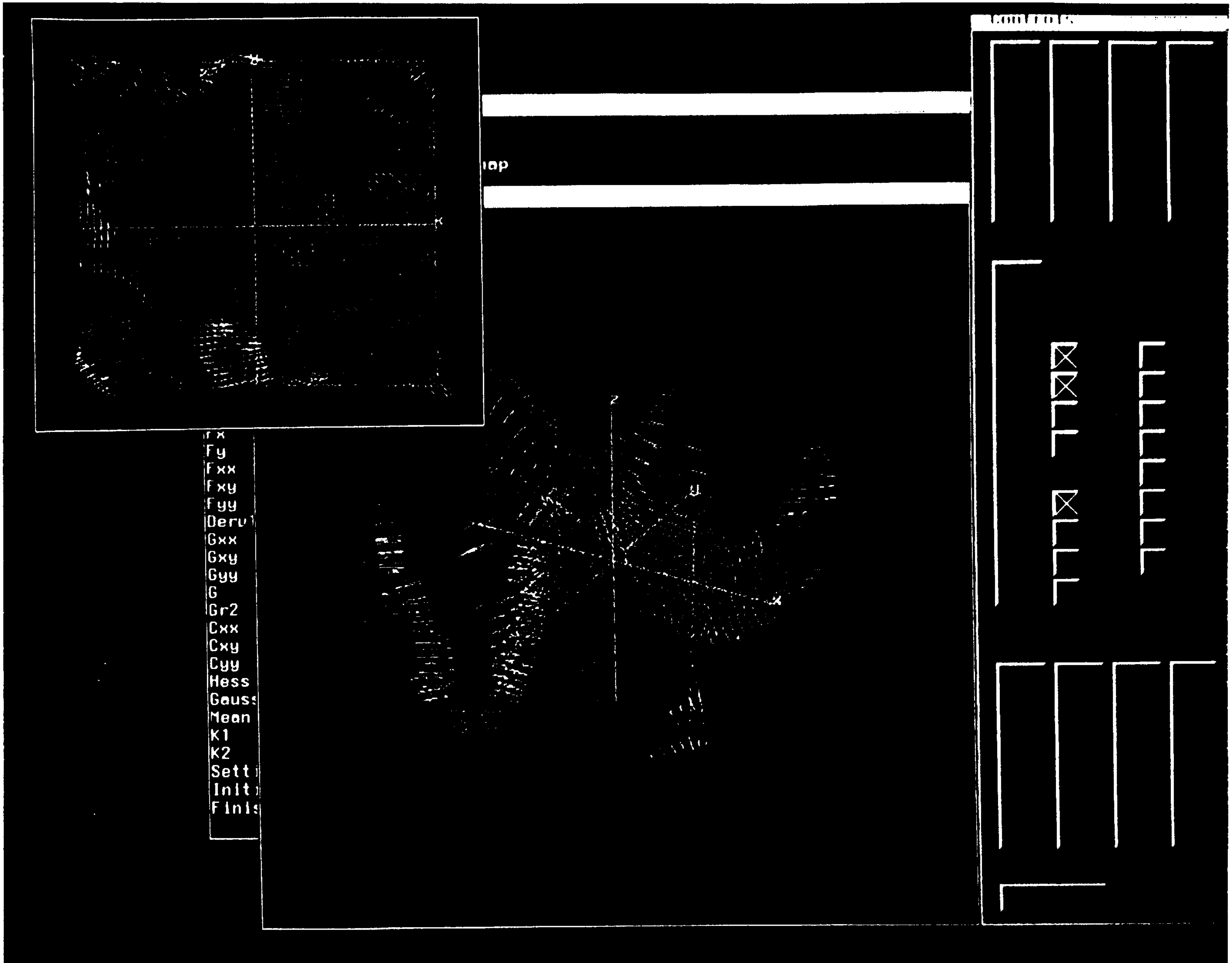
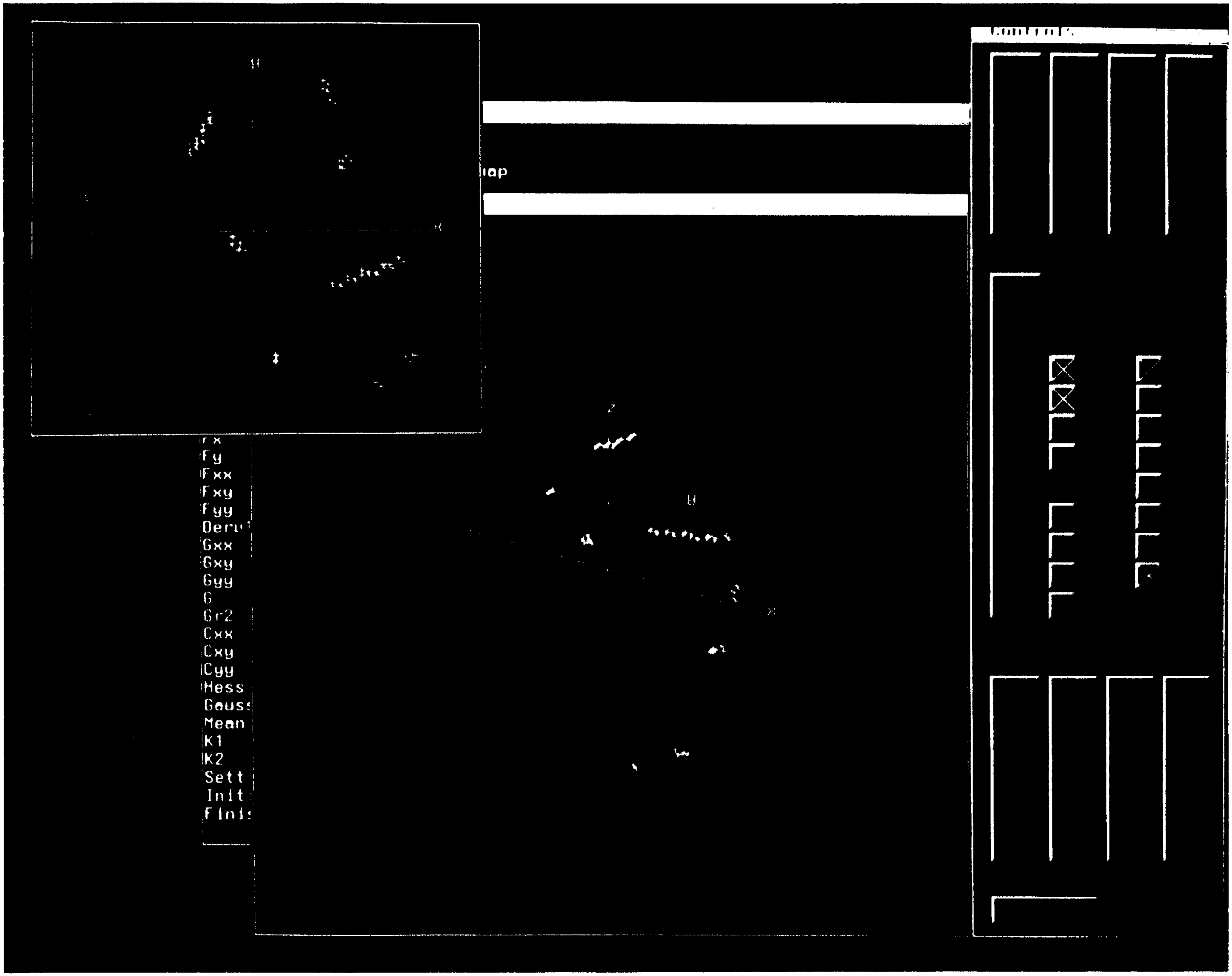
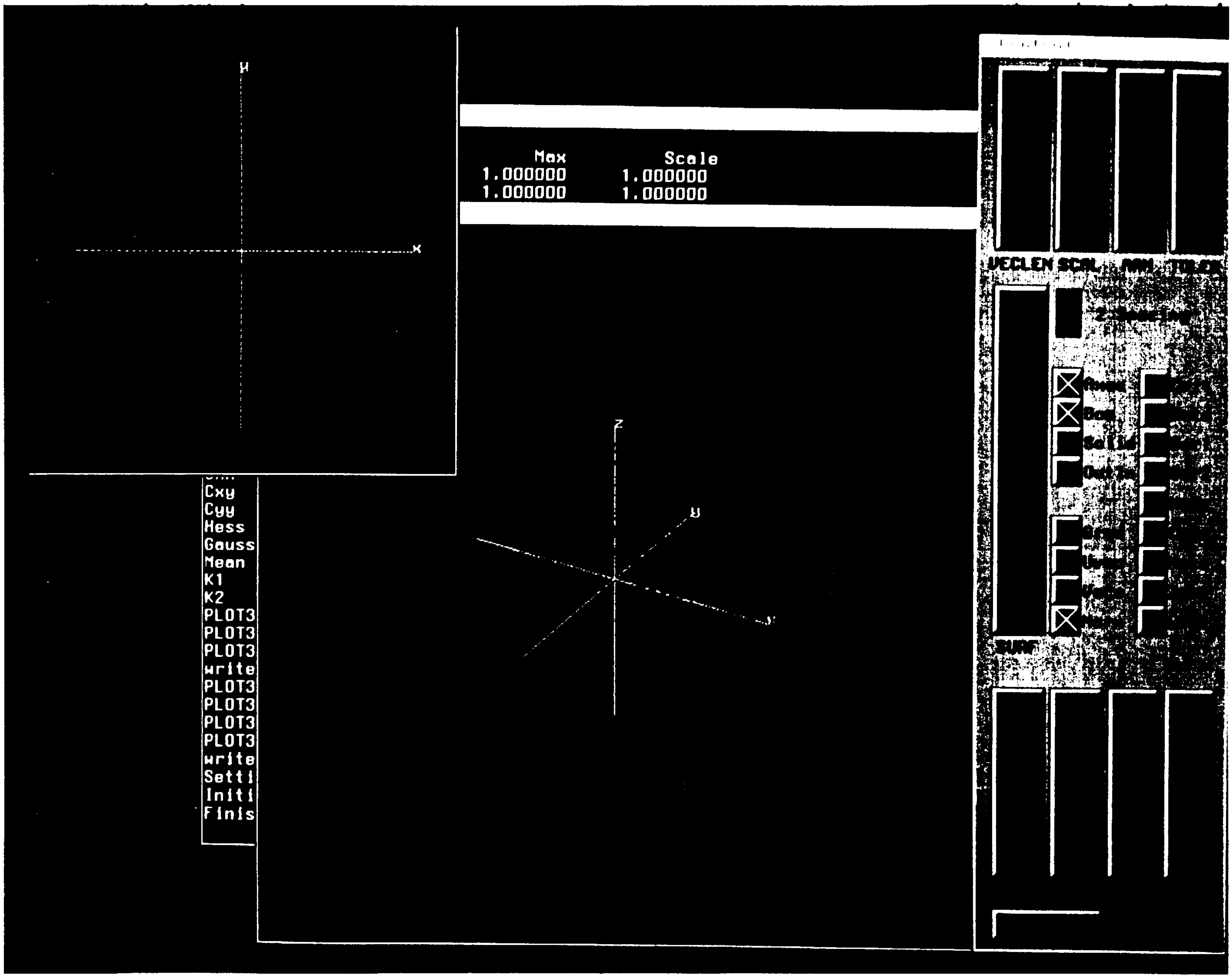
Gxx
Gxy
Gyy
G
Gr2
Cxx
Cxy
Cyy
Hess
Mean
K1
K2

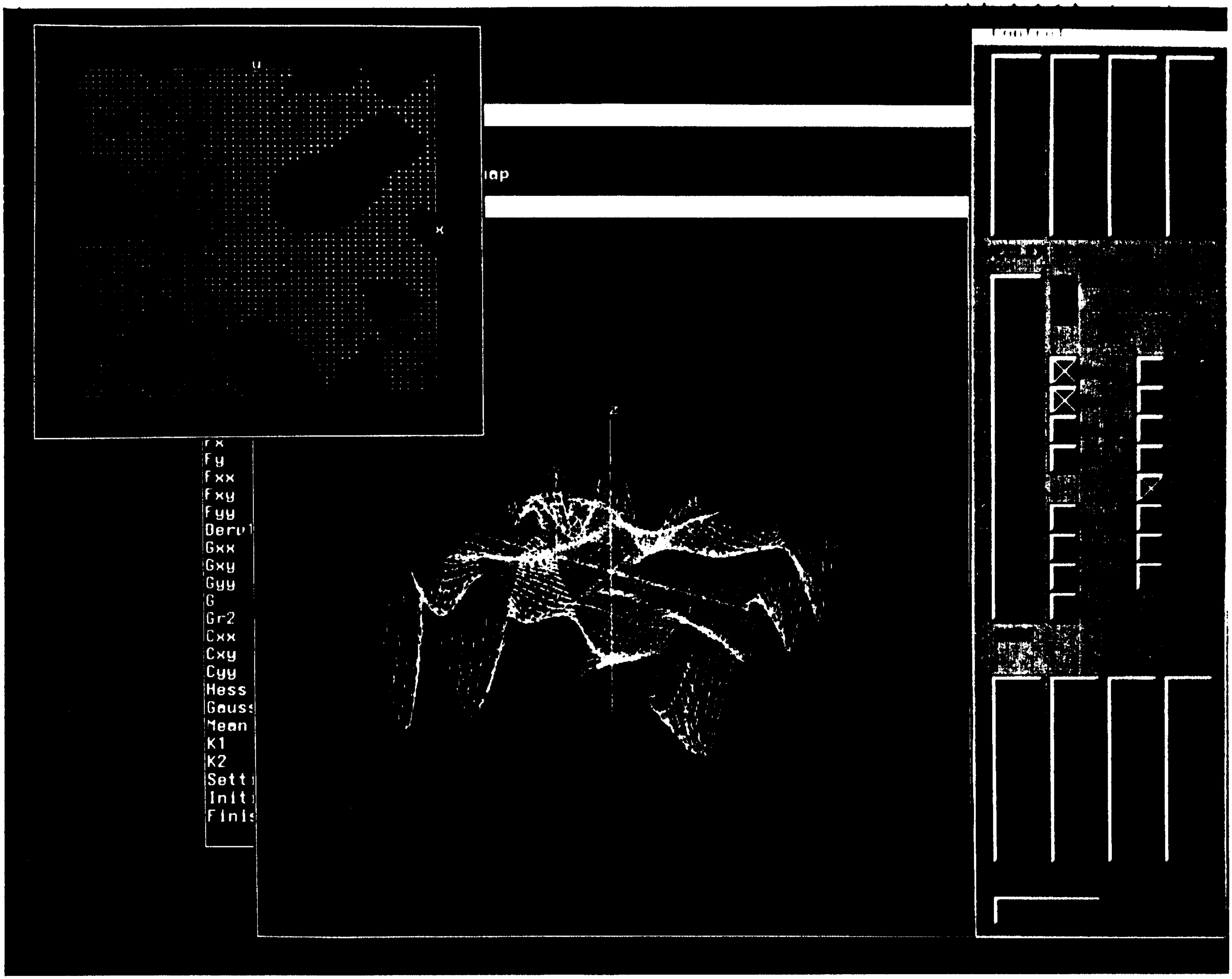


fcrit2 gradient colored by gradient magnitude KERLICK FCRIT2 PLATE 3

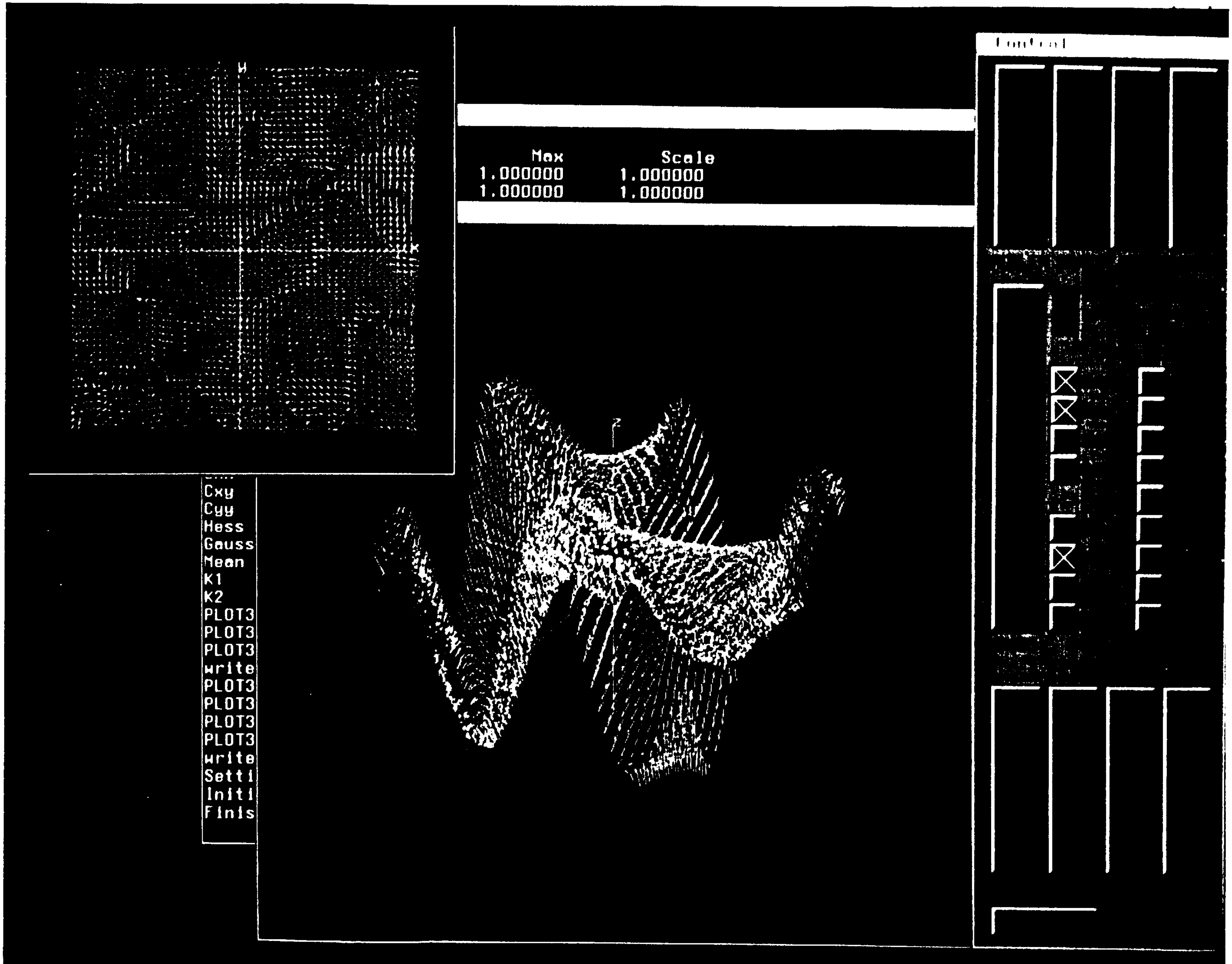

Fy
Fxx
Fxy
Fyy
Gxx
Gxy
Gyy
G
Gr2
Cxx
Cxy
Cyy
Hess
Mean
K1
K2
Sett
Init



fcrit2 surface normals and projections KERLICK FCRIT2 PLATE5

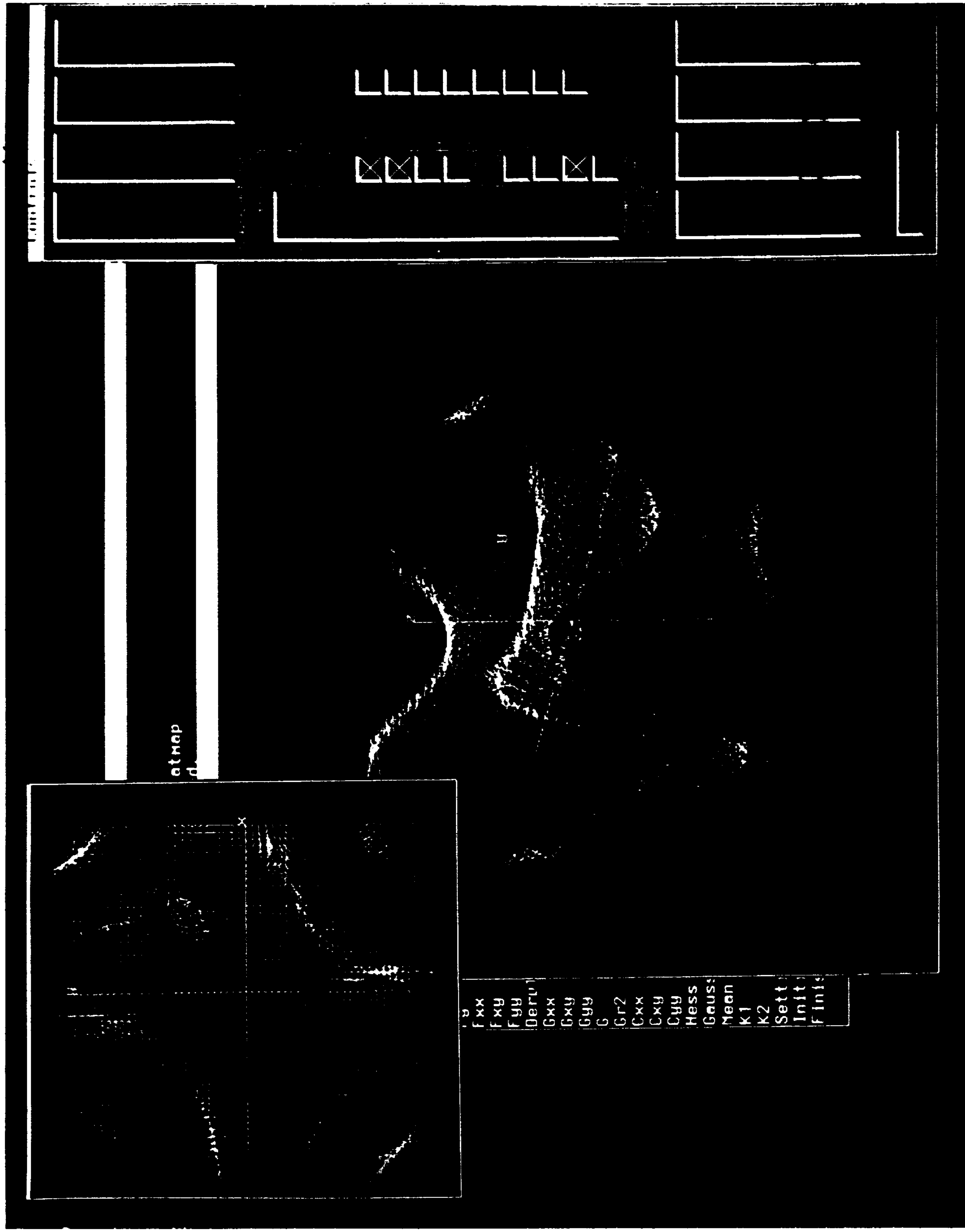


lap
Fy
Fxx
Fxy
Fyy
Deriv
Gxx
Gxy
Gyy
G
Gr2
Cxx
Cxy
Cyy
Hess
Gauss
Mean
K1
K2
Sett
Init
Finis

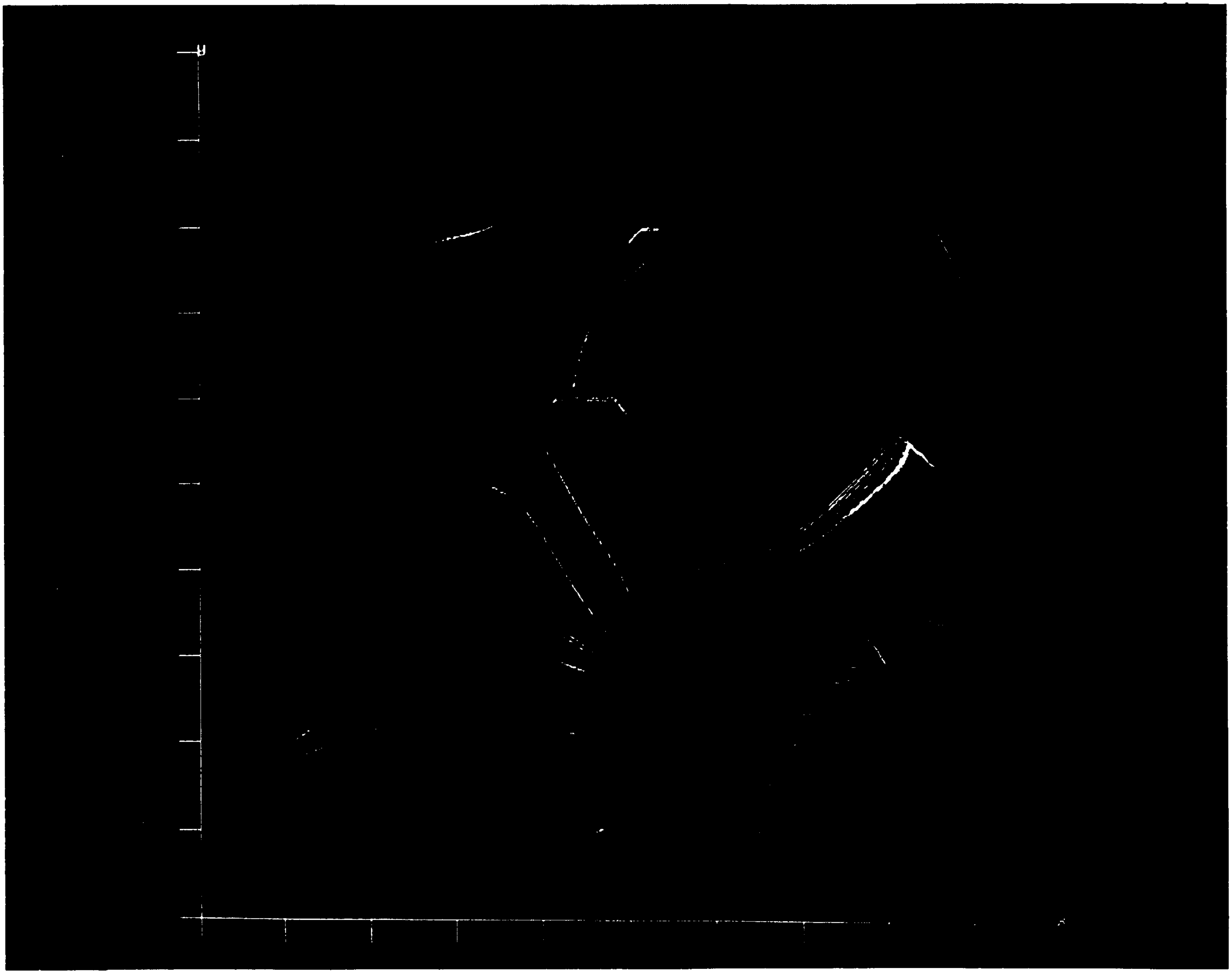


fcrit2 unit curvature eigenvectors

KERLICK FCRIT2 plate 7



atmap
Fxx
Fxy
Fyy
Gxx
Gxy
Gyy
G
Gr2
Cxx
Cxy
Cyy
Hess
Mean
K1
K2



lines of curvature 1st prin dir ridges KERLICK FCRITZ plate 9